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LETTER TO THE EDITOR

A dimensional derivation of the gravitational PCAC correction

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Abstract. The gravitational contribution to the anomaly in the axial current divergence, i.e. $RR^*/384\pi^2$, is derived via dimensional continuation.

The contribution of the gravitational field to the axial current anomaly (Kimura 1969, Delbourgo and Salam 1972) has assumed a new importance in the context of gravitational pseudoparticles (Belavin and Burlankov 1976, Eguchi and Freund 1976, Hawking 1977) and because of its possible effect on the topology of Yang–Mills fields (Charap and Duff 1977). The derivations of the gravitational part of the anomaly have in the past been obtained on the basis of regulator fields or through a point-splitting procedure. Now that dimensional regularisation has become the method of choice for dealing with formally divergent integrals in gauge theories, it is probably worthwhile to rederive the anomaly by this technique. We do so below.

The vector field contributions to the anomaly were worked out by dimensional continuation some time ago (Akyeampong and Delbourgo 1973a, b) and the procedures used there can guide us in our determination of the gravitational contributions. The idea is simply to continue the field theory of spinors, vectors and tensors to arbitrary dimension $n=2l$, and to identify axial vectors and pseudoscalars as three-fold or four-fold antisymmetric tensors respectively. The axial anomaly is then simply given in terms of expectation values of the evanescent current

$$\bar{\psi}\{\Gamma\,.\,\overleftrightarrow{\partial},\Gamma_{[\kappa\lambda\mu\nu]}\}\psi \tag{1}$$

which vanishes in four dimensions at the classical level but can leave its imprint through fermion loop integrals which diverge in four dimensions at the quantum level; the product of the disappearing trace, proportional to $(l-2)$, and the divergence pole $(l-2)^{-1}$ leaves the finite anomaly.

Clearly we must set up general relativity for fermions in arbitrary space–time dimensions in preparation for continuation to $l=2$. This is straightforwardly accomplished in terms of a *vierbein* field L^{α}_{μ} by simply generalising the normal four-dimensional procedure. The relevant part of the Lagrangian is then

$$\mathscr{L}(\psi,L)=(\det L)^{-1}[\tfrac{1}{2}\mathrm{i}L^{\mu\alpha}(\bar{\psi}\Gamma_{\alpha}\overleftrightarrow{\partial}_{\mu}\psi)-m\bar{\psi}\psi+\tfrac{1}{8}\bar{\psi}\{\Gamma^{\alpha},\Gamma^{[\beta\gamma]}\}\psi B_{\alpha\beta\gamma}] \tag{2}$$

with

$$B_{\alpha\beta\gamma}\equiv L^{\mu}_{\alpha}L^{\nu}_{\beta}(L_{\gamma\rho}\Gamma^{\rho}_{\mu\nu}-\partial_{\mu}L_{\nu\gamma})$$

$$\Gamma^{\rho}_{\mu\nu}=\tfrac{1}{2}g^{\rho\lambda}(\partial_{\nu}g_{\lambda\mu}+\partial_{\mu}g_{\lambda\nu}-\partial_{\lambda}g_{\mu\nu})$$

$$g_{\mu\nu}=\eta^{\alpha\beta}L_{\alpha\mu}L_{\beta\nu}.$$

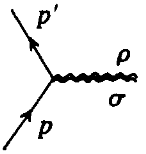
The $\Gamma^{[\alpha_1\cdots\alpha_r]}$ denote r-fold antisymmetric products of the basic Γ matrices associated with the flat-space limit and suitably normalised. We may define the quantum graviton field f through the split $L^{\mu}_{\alpha}=\delta^{\mu}_{\alpha}+\kappa f^{\mu}_{\alpha}$ and thereby reduce our Lagrangian to the form

$$\begin{aligned}\mathscr{L}(\psi, L)=&\{1-\kappa\ \mathrm{Tr}\, f+\tfrac{1}{2}\kappa^2[(\mathrm{Tr}\, f)^2+(\mathrm{Tr}\, f^2)]+\ldots\}\\ &\times[\tfrac{1}{2}\mathrm{i}\bar{\psi}\Gamma\,.\,\overleftrightarrow{\partial}\psi-m\bar{\psi}\psi+\tfrac{1}{2}\mathrm{i}\kappa f^{\alpha\beta}\bar{\psi}\Gamma_{\alpha}\overleftrightarrow{\partial}_{\beta}\psi+\tfrac{1}{16}\kappa^2\bar{\psi}\Gamma^{[\alpha\beta\gamma]}\psi f_{\beta\delta}\overleftrightarrow{\partial}_{\alpha}f^{\delta}{}_{\gamma}+\mathrm{O}(\kappa^3)].\end{aligned}$$

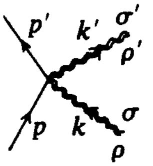
As we are going to treat gravity classically, we shall impose the mass shell conditions $\partial^{\alpha}f_{\alpha\beta}=f_{\gamma\gamma}=0$ to obtain the *effective* interaction Lagrangian

$$\mathscr{L}_{\mathrm{int}}(\psi, L)=\tfrac{1}{2}\mathrm{i}\kappa f^{\alpha\beta}\bar{\psi}\Gamma_{\alpha}\overleftrightarrow{\partial}_{\beta}\psi+\tfrac{1}{2}\kappa^2 f^{2\alpha}{}_{\alpha}\bar{\psi}(\tfrac{1}{2}\mathrm{i}\Gamma\,.\,\overleftrightarrow{\partial}-m)\psi-\tfrac{1}{16}\kappa^2 f_{\beta\delta}\overleftrightarrow{\partial}_{\alpha}f^{\delta}{}_{\gamma}\bar{\psi}\Gamma^{[\alpha\beta\gamma]}\psi+\mathrm{O}(\kappa^3) \quad (3)$$

and hence the *effective* Feynman rules shown in figure 1.



$$\tfrac{1}{4}\kappa[(p+p')_{\rho}\Gamma_{\sigma}+(p+p')_{\sigma}\Gamma_{\rho}]$$



$$\begin{aligned}&\tfrac{1}{2}\kappa^2(\eta_{\rho\rho'}\eta_{\sigma\sigma'}+\eta_{\rho\sigma'}\eta_{\sigma\rho'})[\tfrac{1}{2}\Gamma\,.\,(p+p')-m]\\ &\quad+\tfrac{1}{16}\kappa^2(\eta_{\rho\rho'}\delta^{\lambda}_{\sigma}\delta^{\mu}_{\sigma'}+\eta_{\sigma\sigma'}\delta^{\lambda}_{\rho}\delta^{\mu}_{\rho'}+\eta_{\rho\sigma'}\delta^{\lambda}_{\sigma}\delta^{\mu}_{\rho'}+\eta_{\sigma\rho'}\delta^{\lambda}_{\rho}\delta^{\mu}_{\sigma'})(k+k')^{\nu}\Gamma_{[\lambda\mu\nu]}\end{aligned}$$

Figure 1. Effective spinor graviton couplings to order κ^2.

The pesuodoscalar–two-graviton amplitude, subject to gauge invariance and mass shell conditions, can be decomposed as follows:

$$F^{(\rho_1\sigma_1)(\rho_2\sigma_2)}_{[\kappa\lambda\mu\nu]}(k_1k_2k_3)=P(k_3^2)\delta^{\sigma_1}_{[\kappa}\delta^{\sigma_2}_{\lambda}k_{1\mu}k_{2\nu]}(\eta^{\rho_1\rho_2}k_1\,.\,k_2-k_1{}^{\rho_2}k_2{}^{\rho_1})+(\rho\leftrightarrow\sigma\text{ perms}) \quad (4)$$

where P is a scalar invariant of k_3^2 only (remember $k_1^2=k_2^2=0$). It automatically satisfies $k_{1\rho_1}F^{\rho_1\cdots}_{\cdots}=k_{2\rho_2}F^{\cdot\cdot\rho_2\cdot}_{\cdots}=0$ from the antisymmetrical structure on the right of (3). Likewise the axial–two-graviton amplitude reads

$$F^{(\rho_1\sigma_1)(\rho_2\sigma_2)}_{[\kappa\lambda\mu]}=A(k_3^2)\delta^{\rho_1\rho_2\pi\tau}_{[\kappa\lambda\mu\nu]}k_{1\pi}k_{2\tau}k_3{}^{\nu}(\eta^{\sigma_1\sigma_2}k_1\,.\,k_2-k_1^{\delta_2}k_2^{\sigma_1})+(\rho\leftrightarrow\sigma\text{ perms})$$

and satisfies the requisite constraints. The axial anomaly itself arises from the diagrams of figure 2. Now the beauty of the dimensional technique is that the individual axial A and pseudoscalar P form factors need not be separately evaluated; rather, the anomaly is immediately obtained by insertion of the evanescent vertex $\{\Gamma\,.\,p, \Gamma_{[\kappa\lambda\mu\nu]}\}$ at the meson leg and extracting the appropriate 'divergent' parts of the diagrams.

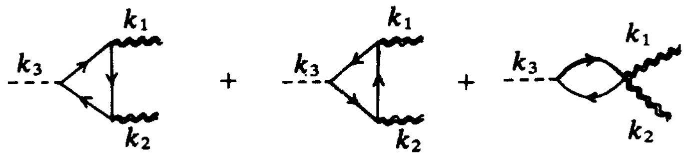


Figure 2. Gravitational contributions to the axial anomaly.

Thus we need only evaluate the loop integrals†

$$\mathrm{Tr}\int \bar{\mathrm{d}}^{2l}p\{\Gamma\,.\,p,\Gamma_{\kappa\lambda\mu\nu}\}[\Gamma\,.\,(p+k_1)-m]^{-1}p^{\rho_1}\Gamma^{\sigma_1}(\Gamma\,.\,p-m)^{-1}p^{\rho_2}\Gamma^{\sigma_2}[\Gamma\,.\,(p-k_2)-m]^{-1}$$

and

$$\mathrm{Tr}\int \bar{\mathrm{d}}^{2l}p\{\Gamma\,.\,p,\Gamma_{\kappa\lambda\mu\nu}\}\eta^{\sigma_1\sigma_2}\delta_\xi^{\rho_1}\delta_\eta^{\rho_2}(k_1-k_2)_\zeta[\Gamma\,.\,(p+k_1)-m]^{-1}\Gamma^{[\xi\eta\tau]}[\Gamma\,.\,(p-k_2)-m]^{-1}.$$

Only the first integral is relevant if we concentrate on the kinematic piece $\delta_{[\kappa}^{\sigma_1}\delta_\lambda^{\sigma_2}k_{1\mu}k_{2\nu]}k_2^{\rho_1}k_1^{\rho_2}$ of (4). By introducing Feynman parameters, the triangle anomaly is thereby reduced to a consideration‡ of

$$\begin{aligned}
2\int_0^1 &\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\delta(1-x-y-z)\int \bar{\mathrm{d}}^{2l}p(p^2+k_3^2xy-m^2)^{-3}\\
&\times\mathrm{Tr}\{\{\Gamma\,.\,p,\Gamma_{\kappa\lambda\mu\nu}\}[\Gamma\,.\,(p+zk_1-xk_3)+m](p+xk_2)^{\rho_1}\Gamma^{\sigma_1}\\
&\times[\Gamma\,.\,(p+xk_2-yk_1)+m](p-yk_1)^{\rho_2}\Gamma^{\sigma_2}[\Gamma\,.\,(p+yk_3-zk_2)+m]\}\\
&=2k_2^{\rho_1}k_1^{\rho_2}\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\theta(1-x-y)xy\\
&\times\int \bar{\mathrm{d}}^{2l}p(p^2+k_3^2xy-m^2)^{-3}\,\mathrm{Tr}(\{\Gamma\,.\,p,\Gamma_{\kappa\lambda\mu\nu}\}\Gamma\,.\,p\Gamma^{\sigma_1}\Gamma^{\sigma_2}\Gamma\,.\,k_1\Gamma\,.\,k_2)+\dots\\
&=2k_2^{\rho_1}k_1^{\rho_2}\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\theta(1-x-y)xy\\
&\times\int \bar{\mathrm{d}}^{2l}p\,p^2(p^2+k_3^2xy-m^2)^{-3}\,.\,2\left(\frac{l-2}{l}\right).\,\mathrm{Tr}(\Gamma_{\kappa\lambda\mu\nu}\Gamma^{\sigma_1}\Gamma^{\sigma_2}\Gamma\,.\,k_1\Gamma\,.\,k_2)+\dots\\
&=2^{l+1}(l-2)k_2^{\rho_1}k_1^{\rho_2}\delta_{[\kappa}^{\sigma_1}\delta_\lambda^{\sigma_2}k_{1\mu}k_{2\nu]}(4\pi)^{-l}\Gamma(2-l)\\
&\times\int (k_3^2xy-m^2)^{l-2}\mathrm{i}xy\theta(1-x-y)\,\mathrm{d}x\,\mathrm{d}y\\
&\xrightarrow[l\to 2]{}-\mathrm{i}\frac{k_2^{\rho_1}k_1^{\rho_2}}{12\pi^2}\delta_{[\kappa}^{\sigma_1}\delta_\lambda^{\sigma_2}k_{1\mu}k_{2\nu]}+\dots.
\end{aligned}$$

Inserting the appropriate numerical factor $\frac{1}{16}\kappa^2$ and the various index permutations, the result may be interpreted as the generally covariant anomaly

$$R_{\kappa\lambda\rho\sigma}R_{\mu\nu}{}^{\rho\sigma}\epsilon^{\kappa\lambda\mu\nu}/384\pi^2.$$

The derivation is therefore really quite easy despite the profusion of indices.

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† Remember that we are able to drop all $\eta_{\rho_1\sigma_1}$, $k_{1\rho_1}$, $k_{2\rho_2}$ terms etc for real gravitons.

‡ Divergent numerators of order p^4 have identically zero trace and are safely zero; we can also discard all convergent numerators of order p^0 for purposes of the anomaly.

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